Mean, Odd Sequential and Triangular Sum Graphs

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ABSTRACT
In this paper, we prove that all odd sequential graphs are mean graphs, but not all mean graphs are an odd sequential graph. We show that some new families generated by some graph operations on some standard graphs are admitting mean labeling and odd sequential labeling. Finally, we conclude some new results in triangular sum graphs.

Keywords
Mean labeling, odd sequential labeling, and Triangular sum graph.

1. INTRODUCTION
A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions [1]. Furthermore, it plays a major role in many numbers of applications such as coding theory, x-ray crystallography, radar, computer technology, astronomy, circuit design, communication networks and database management. This paper will discuss three types of labeling.

First, the mean labeling that was introduced by Somasundaram and Ponraj [7]. They define a function \( f \) as a mean labeling of graph \( G \) if \( f: V(G) \rightarrow \{0, 1, 2, \ldots, q\} \) is injective such that the induced function \( f^*: E(G) \rightarrow \{1, 2, \ldots, q\} \) is bijective. The graph which admits mean labeling is called a mean graph.

Through the last years, some results were introduced in the mean labeling. Firstly, Seoud and Salim [5] presented mean and non-mean graphs of order at most six. They gave an upper bound for the number of edges of a graph with a certain number of vertices to be a mean graph. After that, they constructed families of mean graphs depending on other mean and non-mean graphs. Somasundaram and Ponraj [7, 8, 9, 10] proved the following: \( C_n, K_{2,n}, P_6 \) are mean graphs for any \( m, n \in \mathbb{N} \) but \( W_5 \) is not a mean graph for \( n \geq 3 \). Finally, Seoud and Salim [6] also proved that every odd mean graphs are k-odd mean graphs and put two necessary conditions on the graph to be an odd mean graph. They determined all odd mean graphs of order \( \leq 6 \).

Then odd sequential graph was presented by Singh and Varkey as a graph \( G = (V(G), E(G)) \) with \( p \) vertices and \( q \) edges if there is an injection \( f: V(G) \rightarrow \{0, 1, 2, \ldots, q\} \) and each edge \( xy \) is assigned the label \( f(x) + f(y) \), such that the resulting edge labels are 1, 3, \ldots, 2q-1. The graph which admits odd sequential labeling is known as an odd sequential graph. They proved that Combs and Stars are odd sequential graphs but odd cycles do not admit odd sequential labeling.

The third type of labeling was introduced by S. Hegde and P. Shankaran [3]. They defined the triangular number as the number obtained by adding all positive integers less than or equal to a given positive integer \( n \). Let the \( n \)th triangular number be denoted by \( T_n \), then \( T_n = \frac{n(n+1)}{2} \). Then they use the triangular number to define a triangular sum labeling of a graph \( G \) as a one-to-one function \( f: V(G) \rightarrow N \) (where \( N \) is the set of all non-negative integers) that induces a bijection \( f^*: E(G) \rightarrow \{T_1, T_2, \ldots, T_n\} \) of the edges of \( G \) defined by \( f^*(uv) = f(u) + f(v) \). The graph which admits such a labeling is called a triangular sum graph. Seoud and Salim [4] proved the following results are triangular sum: \( P_n \cup P_m \) symmetrical trees, the graph obtained by identifying the centers of any number of stars, and all trees of order at most 9. The following definitions are used in this paper, all other standard terminologies and notions follow Harary [2].

Definition 1.1. Grid graph can be defined as the Cartesian product of two paths \( P_n \times P_m \).

Definition 1.2. The triangular ladder is a graph obtained from the ladder \( L_n \) by adding the edges \( u_iu_{i+1} \) for \( 1 \leq i \leq n-1 \) and is denoted by \( TL_n \).

Definition 1.3. The graph \( B_{n,2} \) is a bistar obtained from two disjoint copies of \( K_{1,2} \) by joining the center vertices by an edge.

Definition 1.4. The shadow graph of \( G \) is donated by \( D_2(G) \) and is obtained by taking two copies of the graph \( G \) like \( G_1 \) and \( G_2 \) then join each vertex \( u_i \) in \( G_1 \) to the neighbors of the corresponding vertex \( u_i \) in \( G_2 \).

Definition 1.5. The splitting graph of a graph \( G \) is obtained by adding to each vertex \( u \) in \( V(G) \) a new vertex \( v \) such that \( v \) is adjacent to every vertex which is adjacent to \( u \) and is denoted by \( SPL(G) \).

Definition 1.6. The graph \( [P_2; S_m] \) is defined as the graph obtained from \( 2n \) copies of the star \( S_m \) and the path \( P_2 \) with vertices \( u_1, u_2, u_3, \ldots, u_{2n} \) by joining the vertex \( u_i \) with the vertex \( v_j \) of the \( j \)th copy of star \( S_m \) by means of an edge, for some \( 1 \leq j \leq 2n \).

Definition 1.7. The graph \( [P_n; C_3] \) is defined as the graph obtained from \( m \) copies of the cycle \( C_3 \) with vertices \( u_{1j}, u_{2j}, \ldots, u_{nj} \).
The m-stars graph is the graph obtained from a star $S_n$ by identifying each pendant vertex of $S_n$ with an end vertex of the path $P_m$.

**Definition 1.8.** Let $G = (V(G), E(G))$ be a graph and $G_1, G_2, \ldots, G_n$ be $n$ copies of graph $G$, then the graph obtained by adding an edge between $G_i$ and $G_{i+1}$, for $i = 1, 2, \ldots, n-1$ is called the path union of $G$.

**2. GENERAL RESULTS IN MEAN AND ODD SEQUENTIAL GRAPHS**

**Theorem 2.1.** All odd sequential graphs are mean graphs.

**Proof:** Suppose that $G$ is a graph with $p$-vertices and $q$-edges, and $f$ is an injective function defined as $f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}$. For any odd sequential graph $f(v) + f(u) = 2s - 1$, $1 \leq s \leq q$. The edge labels will be $1, 2, 3, \ldots, q$ and we obtain a mean labeling for the graph $G$.

**Remark 2.1** Not all mean graphs are odd sequential graphs. According to Gallian survey [1], Singh and Varkey proved that odd cycles do not admit odd sequential labeling. A mean labeling for the odd cycle $C_7$ could be shown in Fig 1.

![Fig 1: The mean labeling for odd cycle $C_7$.](image)

**Theorem 2.2** The path union for any odd sequential graph $G$ is odd sequential graph.

**Proof:** Suppose that we have $i$ copies of an odd sequential graph $G$, the path union of $G$ is obtained by adding an edge between $G_i$ and $G_{i+1}$ as shown in Fig 2. Let $q$ represent the number of edges in each copy and $u_1, u_2, u_3, \ldots, u_q$ are the vertices of $G$. If $f$ is the function of odd sequential labeling of $G$. Then the vertex labeling in the first copy $G_1$ will follow the function $f$ but the vertex labeling $f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ of the next copies will be as follows:

For $i$ is even:

$f(u_{ij}) = [iq + (i-1)] \cdot f(u_{ij}), 2 \leq i \leq m, 1 \leq j \leq p$

For $i$ is odd and $i \neq 1$:

$f(u_{ij}) = [(q+1)(i-1)] + f(u_{ij}), 3 \leq i \leq m, 1 \leq j \leq p$

![Fig 2: The path union of graph $G$.](image)

**3. NEW RESULTS IN MEAN AND ODD SEQUENTIAL GRAPHS**

**Result 3.1** The path union graph for $i$ copies of cycle $C_n$ and $n \equiv 0 \pmod{4}$ is mean and odd sequential graph.

**Proof:** Consider that we have $i$ copies of even cycle $C_n$ and $n \equiv 0 \pmod{4}$ with vertex set $\{u_i: 1 \leq i \leq m, 1 \leq j \leq n\}$. Suppose that each copy of the cycle $C_n$ is joined to another cycle by an edge. Let $G$ be the path union graph for $i$ copies of even cycle $C_n$, as shown in Fig 3 with $|V(G)| = mn$ and $|E(G)| = mn(n+1)/2$. The vertex labeling $f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ as follows:

For $i = 1$:

$f(u_{11}) = 0$

$f(u_{12}) = 1$

$f(u_{1j}) = n+3j, 3 \leq j \leq (n/2)+2$

For $i$ is even:

If $i \neq 1$:

$f(u_{ij}) = [iq + (i-1)] \cdot f(u_{ij}), 2 \leq i \leq m, 1 \leq j \leq n$

For $i$ is odd and $i \neq 1$:

$f(u_{ij}) = [(q+1)(i-1)] + f(u_{ij}), 3 \leq i \leq m, 1 \leq j \leq n$

**Example 3.1** In Fig 4, we have the mean and odd sequential labeling for the path union of $4$ copies of cycle $C_n$.

![Fig 3: The path union of $i$ copies of cycle $C_n$.](image)
Fig 4: The path union of 4 copies of cycle $C_8$ with mean and odd sequential labeling.

Result 3.2. The path union for $i$ copies of the grid graph $P_m \times P_n$ is a mean and odd sequential graph.

Proof: Consider that we have $i$ copies from grid graph $P_m \times P_n$ with vertex set $\{u_{rs} : 1 \leq r \leq m, 1 \leq s \leq n\}$ and $q$ is the number of edge in each copy. Let $G$ be the graph that represent the path union for $i$ copies of the grid graph $P_m \times P_n$ as shown in Fig 5 with $|V(G)|=i(m+n)$ and $|E(G)|=i(2mn-n)+i-1)$ then we define the vertex labeling $f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ as follows:

For $i = 1$

$f(u_{11}) = r - 1, 1 \leq r \leq m$

$f(u_{1r}) = (2m-1)+f(u_{1,r-1}), 1 \leq r \leq m, 2 \leq s \leq n$

For $i$ is even

$f(u_{ir}) = [iq+(i-1)]f(u_{1r}), 2 \leq i \leq m, 1 \leq r \leq m, 1 \leq s \leq n$

For $i$ is odd and $i \neq 1$

$f(u_{ir}) = [(q+1)+f(u_{i-1})], 3 \leq i \leq m, 1 \leq r \leq m, 1 \leq s \leq n$

Example 3.2. In Fig 6, we have the mean and odd sequential labeling of the path union for 4-copies of the grid $P_4 \times P_3$.

Theorem 3.1. The graph $[P_{2n};S_m]$ is mean and odd sequential graph for $m \geq 3$.

Proof: Consider that we have $2n$ copies of the star $S_m$ with vertex set $\{v_{ij} : 0 \leq i \leq m, 1 \leq j \leq 2n\}$. Let $u_1, u_2, u_3, \ldots, u_m$ be the vertices of the path $P_m$ and $G$ represent the graph $[P_{2n};S_m]$ as shown in Fig 7 with $|V(G)|=2n(m+2)$ and $|E(G)|=2mn+4n-1$. Then, we define the vertex labeling $f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ as follows:

Fig 5: The path union for $i$ copies of the grid graph $P_m \times P_n$.

Fig 6: The path union for 4-copies from grid $P_4 \times P_3$ with mean and odd sequential labeling.
\[ f(u_{j+1}) = j(2m+4), \quad 0 \leq j \leq n-1 \]
\[ f(u_0) = j(2m+4) - 1, \quad 1 \leq j \leq n \]
\[ f(v_{i,j+1}) = j(2m+4) + 1, \quad 0 \leq j \leq n-1 \]
\[ f(v_{i,0}) = j(2m+4) - 2, \quad 1 \leq j \leq n \]
\[ f(v_{i,j}) = j(2m+4) + 2i, \quad 0 \leq j \leq n-1, \quad 1 \leq i \leq m \]
\[ f(v_{i,0}) = (j-1)(2m+4) + 2i + 1, \quad 1 \leq j \leq n, \quad 1 \leq i \leq m \]

**Example 3.3.** In Fig 8, we represent the mean and odd sequential labeling of the graph \([P_4; S_3]\).

**Theorem 3.2.** The graph \([P_m; C_n]\) is mean and odd sequential graph for \(n \equiv 0 \pmod{4}\).

**Proof:** Consider that we have \(m\) copies of the cycle \(C_n\) and \(n \equiv 0 \pmod{4}\) with vertex set \(\{u_i: 1 \leq i \leq m, 1 \leq j \leq n\}\). Let \(v_1, v_2, v_3, \ldots, v_m\) be the vertices of the path \(P_m\) and \(G\) represent the graph \([P_m; C_n]\) as shown in Fig 9 with \(|V(G)| = m(n+1)\) and \(|E(G)| = (n+2) \cdot m - 1\). Then we define the vertex labeling \(f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}\) as follows:

For \(1 \leq i \leq m, i\) is odd.

\[ f(v_i) = (n+2)(i-1) \]

\[ f(u_i) = (n+2)(i-1) + 1 \]

\[ f(u_0) = (n+2)(i-1) + 2 \]

**Example 3.4.** In Fig 10, we represent the mean and odd sequential labeling of the graph \([P_4; S_3]\).

**Theorem 3.2.** The graph \([P_m; C_n]\) is mean and odd sequential graph for \(n \equiv 0 \pmod{4}\).

**Proof:** Consider that we have \(m\) copies of the cycle \(C_n\) and \(n \equiv 0 \pmod{4}\) with vertex set \(\{u_i: 1 \leq i \leq m, 1 \leq j \leq n\}\). Let \(v_1, v_2, v_3, \ldots, v_m\) be the vertices of the path \(P_m\) and \(G\) represent the graph \([P_m; C_n]\) as shown in Fig 9 with \(|V(G)| = m(n+1)\) and \(|E(G)| = (n+2) \cdot m - 1\). Then we define the vertex labeling \(f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}\) as follows:

For \(1 \leq i \leq m, i\) is even.

\[ f(v_i) = (n+2)i-1 \]

\[ f(u_i) = (n+2)i - (1+j), \quad 1 \leq j \leq (n/2) \]

\[ f(u_i) = (n+2)i - (3+j), \quad (n/2) + 1 \leq j \leq n, j\) is odd.

\[ f(u_i) = (n+2)i - (1+j), \quad (n/2) + 1 \leq j \leq n, j\) is even.

**Example 3.4.** In Fig 10, we represent the mean and odd sequential labeling of the graph \([P_4; C_3]\).
4. NEW RESULTS IN MEAN GRAPHS

Theorem 4.1. The triangular ladder $TL_n$ is a mean graph.

Proof: Consider that we have the graph $TL_n$ with two path graphs as shown in Fig 11. Let $\{u_i; 1 \leq i \leq n\}$ be the vertex set of the first path and $\{v_i; 1 \leq i \leq n\}$ is the vertex set of the second path. Suppose that $G$ is triangular ladder graph $TL_n$ with $|V(G)| = 2n$ and $|E(G)| = 4n-3$, then we define the vertex labeling $f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ as follows:

$$
\begin{align*}
    f(u_1) &= 4(i-1)+2, \ 1 \leq i \leq n-1 \\
    f(u_n) &= 4n-3 \\
    f(v_i) &= 4(i-1), \ 1 \leq i \leq n
\end{align*}
$$

Example 4.1. In Fig 12, we have the mean labeling of the triangular ladder $TL_7$.

$$
\begin{align*}
    f(u_0) &= 0 \\
    f(v_0) &= 4n+1 \\
    f(u_i) &= 2i, \ 1 \leq i \leq n \\
    f(v_i) &= 2i, \ n+1 \leq i \leq 2n
\end{align*}
$$

Example 4.2. In Fig 14, we have the mean labeling of the graph $B_{n,n}^{(2)}$.

Fig 10: $[P_4; C_3]$ with mean and odd sequential labeling.

Fig 11: The triangular ladder $TL_n$.

Fig 12: $TL_7$ and its mean labeling.

Fig 13: The Bisrar $B_{n,n}^{(2)}$. 

Fig 14: The mean labeling of the graph $B_{n,n}^{(2)}$. 

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Theorem 4.3. The shadow graph of bistar $D_2(B_{n,n})$ is a mean graph.

**Proof:** Consider that we have two copies of bistar $B_{n,n}$ and let the vertex set $\{u_0, w_0, u_i, w_i; 1 \leq i \leq n\}$ corresponding to the set $\{v_0, l_0, v_i, l_i; 1 \leq i \leq n\}$. Let $G$ is the graph $D_2(B_{n,n})$ as shown in Fig 15 with $|V(G)|=4(n+1)$ and $|E(G)|=4(2n+1)$ then we define the vertex labeling $f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ as follows:

- $f(u_0) = 0$
- $f(u_i) = 2i, 1 \leq i \leq n$
- $f(v_0) = 4n$
- $f(v_i) = 2(n+i), 1 \leq i \leq n-1$
- $f(l_0) = 8n+3$
- $f(l_i) = 8(n+1)-4i, 1 \leq i \leq n$

**Example 4.3** In Fig 16, we represent the mean labeling of the shadow graph of $(B_{n,n})$.

- $f(w_0) = 8n+1$
- $f(w_i) = 4n+5$
- $f(l_0) = 8n+1$
- $f(l_i) = 4n+5$
- $f(v_0) = 4n-1$
- $f(v_i) = 4n+i+1, 1 \leq i \leq n-1$

Fig 15: The shadow graph of bistar $D_2(B_{n,n})$.

Fig 16: $D_2(B_{n,n})$ and its mean labeling.
Theorem 4.4. The split graph of bistar SPL \((B_{n,n})\) is a mean graph.

Proof: Consider that we have bistar \(B_{n,n}\) with the vertex set \(\{u_0, v_0, u_i, v_i: 1 \leq i \leq n\}\) where \(u_i, v_i\) are the pendant vertices. To get the split graph of bistar SPL \((B_{n,n})\) we add the vertices \(w_0, l_0, w_i, l_i\) corresponding to \(u_0, v_0, u_i, v_i\) where \(1 \leq i \leq n\). Let \(G\) is the graph SPL \((B_{n,n})\) as shown in Fig 17 with \(|V(G)|=4(n+1)\) and \(|E(G)|=3(2n+1)\) then we define the vertex labeling \(f: V(G) \to \{0, 1, 2, \ldots, q\}\) as follows in two cases:

Case 1: \(n\) is even number.

- \(f(u_0) = 1\)
- \(f(u_i) = 2(n+i-1), 1 \leq i \leq n\)
- \(f(w_0) = 2n+1\)

Case 2: \(n\) is odd number.

- \(f(u_0) = 0\)
- \(f(u_i) = 2(n+i), 1 \leq i \leq n\)
- \(f(w_0) = 2n-1\)
- \(f(w_i) = 2i, 1 \leq i \leq n\)
- \(f(v_0) = 6n+1\)
- \(f(v_i) = 2(3n+1)\)
- \(f(v_i) = 2(3n-2i)+7, 2 \leq i \leq n\)

Example 4.4. In Fig 18, we represent the mean labeling of the split graph of bistar \((B_{n,n})\).

\[f(v_i) = 2(i-1), 1 \leq i \leq n\]
\[f(v) = 6n+1\]
\[f(v_i) = 2(3n+1)\]
\[f(v_i) = 2(3n-2i)+7, 2 \leq i \leq n\]

Fig 17: The split graph of bistar SPL \((B_{n,n})\).

Fig 18: SPL \((B_{8,8})\) and its mean labeling.

Example 4.5. In the following Fig 19, we represent the mean labeling of the split graph of bistar \((B_{7,7})\).
5. THE RELATION BETWEEN MEAN GRAPH, ODD SEQUENTIAL SUM GRAPH AND TRIANGULAR SUM GRAPH

Theorem 5.1: The graphs can admit mean labeling and triangular sum labeling if and only if $|E|\leq 3$.

Proof: Only the three graphs that are shown in Fig 20 are admitting mean labeling and triangular sum labeling. Suppose that $uv$ is the fourth edge that labeled by 4 in any mean graph ($f(u)$ stands for the label of the vertex $u$), this edge is possible only in the following cases:

- $f(u) = 0$ and $f(v) = 8$ are adjacent.
- $f(u) = 1$ and $f(v) = 7$ are adjacent.
- $f(u) = 2$ and $f(v) = 6$ are adjacent.
- $f(u) = 3$ and $f(v) = 5$ are adjacent.
- $f(u) = 0$ and $f(v) = 7$ are adjacent.
- $f(u) = 1$ and $f(v) = 6$ are adjacent.
- $f(u) = 2$ and $f(v) = 5$ are adjacent.
- $f(u) = 3$ and $f(v) = 4$ are adjacent.

However, the fourth edge in any triangular sum graph must be labeled by $T_q=10$, and no one of the previous cases satisfies this condition.

Theorem 5.2: There are two graphs only that admit mean labeling, odd sequential labeling and triangular sum labeling.

Proof: Only the two graphs in Fig 21 admit mean labeling, odd sequential labeling and triangular sum labeling as if the label of the vertex $u$ is odd sequential graph.

6. NEW RESULTS IN TRIANGULAR SUM GRAPH

Lemma 6.1: The two vertices that are labeled by 0 and 1 must be adjacent in any triangular sum graph [11].

Proof: Let $v_1, v_2, v_3, v_4$ be the vertices of a polygon. If the two vertices $v_i, v_j$ are labeled by 0 and 1 respectively then according to lemma 6.1 the two vertices $v_i, v_j$ must be adjacent in any triangular sum graph $G$.

Let $v_j$ be labeled by 2; then the vertex $v_j$ will be adjacent to the vertex $v_2$ (the vertex that is labeled by 1) to obtain the edge label 3. If the vertex $v_j$ is labeled by any $x \in N, x\neq 0, 1, 2$ then, $v_j$ will be adjacent to the vertices $v_{i+1}$ and $v_{i+2}$ and this will give us two edges with labels $T_n$ and $T_{n+2}$ where $n =3, 4, \ldots, q$. However, this contradicts with the definition of triangular sum graph as the absolute difference between the two triangular numbers $T_n$ and $T_{n+2}$ for all $n \geq 3$ should be greater than 2.

Lemma 6.3: For any triangular sum graph $G$, the vertices that are labeled by 0, 1 and 3 cannot be in the same polygon of 4 vertices contained in $G$.

Proof: Let $v_1, v_2, v_3, v_4$ be the vertices of a polygon. If the two vertices $v_i, v_j$ are labeled by 0 and 1 respectively then according to lemma 6.1 the two vertices $v_i, v_j$ must be adjacent in any triangular sum graph $G$.

Let $v_3$ be labeled by 3; then the vertex $v_3$ will be adjacent to the vertex $v_1$ (the vertex that is labeled by 0) to obtain the edge label 3. If the vertex $v_3$ is labeled by any $x \in N, x\neq 0, 1, 3$ then, $v_3$ will be adjacent to the vertices $v_{i+1}$ and $v_{i+2}$ as this will contradict with lemma 6.2. $v_3$ will be adjacent to the vertices $v_2$ and $v_4$ and this will give us two edges with labels $1+3$ and $3+3$. But this contradicts with the definition of triangular sum graph as the difference
between any two triangular numbers $T_n$ for $n \geq 3$ is greater than 2.

**Theorem 6.1:** The bipartite graph $K_{2,n}$ are not triangular sum graphs.

**Proof:** Suppose the bipartite graph $K_{2,n}$. Every edge of graph $K_{2,n}$ is a part of polygon with 4-vertices. According to lemma 6.1 the two vertices that are labeled by 0 and 1 must be adjacent. In order to obtain the edge with label 3 we have two possibilities:

(i) $3=1+2$  
(ii) $3=0+3$  

(i) If $3=1+2$ then the vertices with labels 0, 1 and 2 will be a part of a polygon of 4-vertices and this contradict with Lemma 6.2.

(ii) If $3=0+3$ then the vertices with labels 0, 1 and 3 will be a part of a polygon of 4-vertices and this contradict with Lemma 6.3.

From the two previous cases we conclude that the edge that labeled by 3 cannot exist and this prove that $K_{2,n}$ cannot admitting triangular sum labeling.

**Theorem 6.2:** Among all graphs of order at most 5 only the following graphs in table (6-2) are admitting triangular sum labeling.

Before the proof we will introduce the following lemmas:

**Lemma 6.4:** Every Null graph $N_n$ cannot be triangular sum graph.

**Lemma 6.5:** If $G$ is a triangular sum graph, then $G \cup N_n$ is also triangular sum graph.

**Lemma 6.6:** If $G$ is not a triangular sum graph, then $G \cup N_n$ is also not triangular sum graph.

**Proof:** The following table will introduce all graphs of order at most 5 that do not admit to triangular sum labeling.

| Table 1. Graphs of order at most 5 that are not admitting triangular sum labeling. |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 |
Proof: The graphs (1), (2), (3), (8), (9), (10), (11), (15), (17) and (20) are not triangular sum graphs by Theorem 2.5 due to Vaidya et al. [11]; the two graphs (6) and (26) are not triangular sum graphs by Theorem 6.1. Also, the graphs (33), (34), (35) and (36) are not triangular sum graphs by Lemma 6.4. The two graphs (14) and (24) are not triangular sum graphs by Lemma 2.2 due to Vaidya et al. [11], by Lemma 6.2 and Lemma 6.3.

The graphs (4), (13), (22),(21), and (29) cannot be triangular sum graphs as if the vertex that is not a part of a triangle be labeled by 0. After that, the only way to get the edge with label 3 that the two vertices with labels 1 and 2 are adjacent and this contradict with Lemma 3.3 due to Vaidya, et al. [11]. But if this vertex be labeled by 1 then to get the edge with label 3 the two vertices with labels 0 and 3 will be adjacent and this make the only way to get the edge with label 6 that the two vertices with labels 0 and 6 are adjacent. Consequently, this will give us three edges with labels 2, 3 and 5 which are not triangular numbers.

The graph (5) is not triangular sum graph as the only two ways to get the edge with label 3 that the two vertices withlabels 0 and 3 or 1 and 2 are adjacent. This impossible as the graph is disconnected and the two vertices that are labeled by 0 and 1 must be adjacent by Lemma 6.1.

The graphs (7), (12), (16), (18), (19), (23) and (31) are not triangular sum graphs by Lemma 6.6.

The graph (25) is not triangular sum graph as the two vertices in the path P₂ must be labeled by 0 and 1 by Lemma 2.2 due to Vaidya et al. [11]. Then the only way to get the edge with label 3 that the two vertices with labels 1 and 2 or 0 and 3 are adjacent and this impossible as the graph is disconnected.

To prove that the following graphs: (27), (28), (30) and (32) cannot admit triangular sum labeling; we have some cases for discussion. So we will draw all possible cases for each graph. The two graphs (28) and (30) are not triangular sum graph as by Lemma 2.1, Lemma 2.2 due to Vaidya et al. [11] and Lemma 6.1, the number of cases will reduce to only five cases. These cases shown in Fig 22 and Fig 23 for labeling graph (28) and graph (30) respectively. But for the graph (27) and according to Lemma 6.2 and Lemma 6.3 the number of cases will reduce to four cases shown in Fig 24. Finally, for graph (32) we have three cases shown in Fig 25.
We see clearly that no one of the previous cases in Figs (22), (23), (24) and (25) can admit triangular sum labeling. To complete our proof, the following table will present all graphs of order at most 5 that are admitting triangular sum labeling.

**Table 2. Graphs of order at most 5 that are admitting triangular sum labeling.**

7. NEW FAMILIES OF TRIANGULAR SUM GRAPHS

**Theorem 7.1:** The Bistar graph $B_{n,m}$ is a triangular sum graph.

**Proof:** Let $G$ be the Bistar graph $B_{n,m}$ as shown in Fig 26 with $|V(G)|=m+n+2$ and $|E(G)|=m+n+1$ then we define the vertex labeling $f: V(G) \rightarrow \mathbb{N}$ as follows:

**Fig 26: The Bistar graph $B_{n,m}$**
Example 5.2.1. In the following Fig 27 we have the triangular sum labeling for the Bistar graph $B_{4,5}$.

Fig 27: The Bistar graph $B_{4,5}$ with triangular sum labeling.

Theorem 7.2: The $m$-stars graph is triangular sum graph.

Proof: Let $G$ be the $m$-star graph as shown in Fig 28 with $|V(G)| = mn+1$ and $|E(G)| = mn$, then we define the vertex labeling $f: V(G) \rightarrow \mathbb{N}$ as follows:

$$f(v_0) = 0$$
$$f(u_i) = (i+1)(i+2)/2, \ 1 \leq i \leq n$$
$$f(v_i) = [(i+1)(i+2)/2] - 1, \ n+1 \leq i \leq m$$

Example 5.2.1. In the following Fig 29, we have the triangular sum labeling for the 4-stars graph.

Fig 28: The $m$-stars graph.

8. CONCLUSION

Since graph labeling serve as practically useful model for a wide range of applications. It is desired to have generalized the following results. This paper focused on mean graphs, odd sequential graphs and triangular sum graphs. We proved that all odd sequential graphs are mean graphs and the path union for any odd sequential graph $G$ is odd sequential graph. After that, we presented families of graphs that are admitting mean labeling and odd sequential labeling. Finally, we defined the relation between mean graph, odd sequential graph and triangular sum graph and getting some new results in triangular sum graphs.

9. REFERENCES